Proof of the future value annuity formula

- The basic idea for a future value annuity is that every month we receive compound interest on our new payment along with all of our previous payments. Therefore at each time when we are going to compound the interest, each payment has been compounded an additional time.

- Example: Payment \( R \) is paid four times a year and the interest is compounded quarterly for an annual interest rate of \( r \) for one year:
  
  - Start: 0
  - After quarter one: \( R \)
    
    - After quarter two the original \( R \) is compounded once now and we pay an additional \( R \)
      
      \[
      R \left( 1 + \frac{r}{n} \right)^2 + R \left( 1 + \frac{r}{n} \right) + R
      \]
    
    - After quarter three the original \( R \) has been compounded twice, the second \( R \) has been compounded once, and we pay an additional \( R \)
      
      \[
      R \left( 1 + \frac{r}{n} \right)^3 + R \left( 1 + \frac{r}{n} \right)^2 + R \left( 1 + \frac{r}{n} \right) + R
      \]
    
    - After quarter four (one year) the original \( R \) has been compounded three times, the second \( R \) has been compounded twice, the third \( R \) has been compounded once, and we pay an additional \( R \)
      
      \[
      R \left( 1 + \frac{r}{n} \right)^4 + R \left( 1 + \frac{r}{n} \right)^3 + R \left( 1 + \frac{r}{n} \right)^2 + R \left( 1 + \frac{r}{n} \right) + R
      \]
  
  - Thus you can see the pattern that the original payment will always be compounded one less that compounding period, which in general will be \( nt \) for \( n \) compounds per year and \( t \) total years

- We can now see the general formula for Future Value \( FV \) is
  
  \[
  FV = R \left( 1 + \frac{r}{n} \right)^{nt-1} + R \left( 1 + \frac{r}{n} \right)^{nt-2} + \cdots + R \left( 1 + \frac{r}{n} \right)^1 + R \left( 1 + \frac{r}{n} \right)^0
  \]
  
  - To simplify our numbers, let \( 1 + \frac{r}{n} = x \) and \( nt = y \). Now we have
    
    \[
    = Rx^{y-1} + Rx^{y-2} + \cdots + Rx + R
    \]
  
  - We factor out an \( R \), leaving us with
    
    \[
    = R(x^{y-1} + x^{y-2} + \cdots + x + 1)
    \]
As this is a geometric series, our trick is to multiply by $1 - x$, but as we are not allowed to multiply by random things in an equation, we will also divide by $1 - x$

$$R \left( \frac{(1 - x)(x^{y-1} + x^{y-2} + \cdots + x + 1)}{1 - x} \right)$$

Multiplying the $1 - x$ through we will see many of the same terms

$$R \left( \frac{(x^{y-1} + x^{y-2} + \cdots + x + 1) - (x^y + x^{y-1} + x^{y-2} + \cdots + x)}{1 - x} \right)$$

Since we have the positive and negative of several terms we will let all of those become 0

$$R \frac{1 - x^y}{1 - x}$$

Now we will substitute back in for the $x$ and $y$ with their original values

$$R \frac{1 - \left( \frac{1 + \frac{r}{n} \cdot nt}{1 + \frac{r}{n}} \right)}{1 - \left( \frac{1 + \frac{r}{n} \cdot nt}{1 + \frac{r}{n}} \right)}$$

We can subtract the ones on the bottom leaving us with

$$R \frac{1 - \left( \frac{1 + \frac{r}{n} \cdot nt}{r} \right)}{\frac{r}{n}}$$

Since we have a negative on top and bottom, we will multiply the top and bottom by negative one and rearrange the subtraction on top

$$R \left( \frac{1 + \frac{r}{n} \cdot nt}{r} - 1 \right)$$

Thus we have now proved the future value annuity formula which is

$$FV = R \left( \frac{1 + \frac{r}{n} \cdot nt}{r} - 1 \right)$$
• To prove the present value annuity formula let’s say that our Present Value $PV$ compounded monthly will equal the future value of the amount of the loan payment $R$ because the payments should eventually equal the cost of the actual purchase $PV$ and the interest accumulated each month

$$PV \left(1 + \frac{r}{n}\right)^{nt} = FV$$

$$PV \left(1 + \frac{r}{n}\right)^{nt} = R \left(\frac{\left(1 + \frac{r}{n}\right)^{nt}}{\frac{r}{n}} - 1\right)$$

• To solve for the actual $PV$ we will basically divide the

$$\left(1 + \frac{r}{n}\right)^{nt}$$

to the other side, but to do that we will actually multiply by

$$\left(1 + \frac{r}{n}\right)^{-nt}$$

on each side and then multiply through to simplify:

$$PV \left(1 + \frac{r}{n}\right)^{nt} \cdot \left(1 + \frac{r}{n}\right)^{-nt} = R \left(\frac{\left(1 + \frac{r}{n}\right)^{nt}}{\frac{r}{n}} - 1\right) \cdot \left(1 + \frac{r}{n}\right)^{-nt}$$

$$PV (1 + \frac{r}{n})^{0} = R \left(\frac{\left(1 + \frac{r}{n}\right)^{0}}{\frac{r}{n}} - (1 + \frac{r}{n})^{-nt}\right)$$

$$PV = R \frac{1 - (1 + \frac{r}{n})^{-nt}}{\frac{r}{n}}$$

• Thus we have proved the Present Value annuity formula.